Mathematisches Forschungsinstitut Oberwolfach

Report No. 42/2015

DOI: 10.4171/OWR/2015/42

The Mathematics and Statistics of Quantitative Risk Management

Organised by
Richard Davis, New York
Paul Embrechts, Zürich
Thomas Mikosch, Copenhagen
Andrew Patton, Durham

20 September – 26 September 2015

Abstract. It was the aim of this workshop to gather a multidisclipinary and international group of scientists at the forefront of research in econometrics, financial time series analysis, extreme value theory, financial mathematics, insurance mathematics and quantitative risk management. The heterogeneous composition of this group of researchers allowed one to discuss different facets of the mathematics and statistics of quantitative risk management, to communicate the state-of-the-art in the different areas, and to point towards new directions of research.

Mathematics Subject Classification (2010): Primary: 62, secondary: 60.

Introduction by the Organisers

This meeting was well attended by more than 50 participants with broad geographic representation from five continents. The participants were statisticians, applied probabilists, financial mathematicians, time series analysts, econometricians, and economists. They represented rather different aspects of quantitative risk management.

This meeting was the third of its kind. The first Oberwolfach Meeting on the *Mathematics and Statistics of Quantitative Risk Management,* was held in March 2008, the second one in February 2012. At all these meetings we experienced an active interaction of the participants from rather diverse areas of expertise. They discussed a very wide spectrum of risk related questions.

For the first time, a larger group of econometricians attended this meeting. Naturally, this fact shifted the focus somewhat. The main topics of the talks and discussions were modeling and prediction of prices, volatilities and interest rates using sophisticated, often high-dimensional econometric models (Andersen, Creal, Duan, Engle, Hautsch, Tauchen, Teräsverta, Tsay). Among these, the lecture of the winner of the 2003 Nobel Prize, Robert F. Engle, was a special highlight of the meeting. He addressed how the preceding financial crisis taught lessons to risk managers that one "should account for the risk that the risk can change". He proposed a probability based scenario analysis as a solution to this problem. The talk by Torben Andersen highlighted new derivative securities (so-called "weeklies") which can reveal information about short-term volatility and the likelihood of negative versus positive jumps in prices. Hautsch, Teräsvirta and Tsay presented new methods and models for covariance matrices, and Creal presented a new approach for modeling time-varying parameters in semiparametric models. Zhou reported about progress of his work on the mathematics behind prospect theory and Härdle talked about his contribution to the block chain technology.

A recent well-established field of risk management deals with the estimation of volatility. Some major experts were present (Jacod, Podolskij, Tauchen, Todorov). New topics include the determination of the rank of a spot covariance matrix (Podolskij), testing for time-varying jump activity in asset prices (Todorov), and "regression" methods for studying the dependence between jumps in two or more asset prices (Tauchen).

One of the classical topics of risk analyses is the study of extremes. Various researchers discussed modern developments in extreme value theory and statistics (Blanchet, Cooley, de Carvalho, Janßen, Kabluchko, Klüppelberg, Liu, Nolan, Segers, Strokorb). Current topics in these fields are the modeling and estimation of multivariate and spatio-temporal extremes, the study of suitable random structures for describing spatial and temporal extremal dependence, such as max-stable fields and graphical models.

Another classical risk-related field is insurance mathematics. Although classical risk theory is still under investigation (Albrecher), this area is nowadays supplemented by financial views at solvency capital calculation (Filipović), rare event simulation techniques (Hult) and stochastic optimization methods (Steffensen). A major theoretical and practical problem is longevity (Loisel).

Financial mathematics is the younger brother of insurance mathematics. We listened to talks on problems which are oriented towards real-life problems in the field: simulation of option prices (Rogers), mathematical modeling of financial bubbles (Protter).

Quantitative risk management is concerned with the modeling and estimation of special (e.g. conditional, partial) dependence structures and the study of aggregated risks under dependence (Chavez-Demoulin, Rüschendorf, Wang), the numerical and statistical calculation of risk indicators such as Value-at-Risk, expected shortfall (Hofert, Yuen) and the study of their properties such as elicitability (Ziegel).

This meeting brought together a group of researchers from a variety of fields, who discussed the state-of-the-art of mathematical, econometric and statistical modeling, estimation, numerical and simulation techniques for quantitative risk management. Many discussions were triggered by risk phenomena and problems in the financial, insurance and regulatory domains. The scientific results presented no doubt led to a better understanding of some of the important regulatory issues facing the industry as well as society, and eventually will contribute to solutions to some of the key questions posed.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: The Mathematics and Statistics of Quantitative Risk Management

Table of Contents

Abstracts

Simple identities for randomized observations in risk theory

Hansjörg Albrecher
(joint work with Jevgenijs Ivanovs)

For a continuous-time surplus process of an insurance portfolio over time, we consider the effects of observing the process only at discrete random times on the resulting probabilities to detect first passage over a threshold or first passage below zero, i.e. ruin. When the random observation times are assumed to be epochs of an independent homogeneous Poisson process, it turns out that the respective probabilities are related to the ones of continuous observation through surprisingly simple identities. This holds for general Lévy processes as models for the surplus process. Moreover, we identify a simple link between two-sided exit problems with one continuous and one Poisson exit. Finally, Poisson exit of a reflected surplus process is connected to the continuous exit of a surplus process reflected at Poisson epochs (which has natural interpretations in terms of dividend payments according to horizontal barrier strategies), and a link between some Parisian type exit problems is established. With the appropriate perspective, the proofs of all these relations turn out to be quite elementary. For spectrally one-sided Lévy processes this approach enables alternative and simpler proofs for a number of previously established identities (see e.g. [2]), providing additional insight.

References

[1] H. Albrecher and J. Ivanovs (2015). *Strikingly simple identities relating exit problems for Lévy processes under continuous and Poisson observations.* Working paper, arXiv: 1507.03848.

[2] H. Albrecher, J. Ivanovs, X. Zhou (2015). *Exit identities for Lévy processes observed at Poisson arrival times.* Bernoulli, in press.

Pricing short-term market risk: evidence from weekly options

Torben G. Andersen
(joint work with Nicola Fusari and Viktor Todorov)

We study short-term market risks implied by weekly S&P 500 index options. The introduction of weekly options has dramatically shifted the maturity profile of traded options over the last five years, with a substantial proportion now having expiry within one week. Economically, this reflects a desire among investors for actively managing their exposure to very short-term risks. Such short-dated options provide an easy and direct way to study market volatility and jump risks. Unlike longer-dated options, they are largely insensitive to the risk of intertemporal shifts in the economic environment, i.e., changes in the investment opportunity set. Adopting a novel general semi-nonparametric approach, we uncover variation in the shape of the negative market jump tail risk which is not spanned by market volatility. Incidents of such tail shape shifts coincide with serious mispricing of standard parametric models for longer-dated options. As such, our approach allows for easy identification of periods of heightened concerns about negative tail events on the market that are not always "signaled" by the level of market volatility and elude standard asset pricing models.

References

[1] T.G. Andersen, N. Fusari and V. Todorov (2015). *Parametric inference and dynamic state recovery from option panels.* Econometrica **83**, 1081–1145.
[2] T.G. Andersen, N. Fusari and V. Todorov (2015). *The risk premia embedded in index options.* Journal of Financial Economics **117**, 558–584.

Robust extreme value analysis

Jose Blanchet
(joint work with Karthyek Murthy)

Extreme value theory (EVT) is used in a wide range of quantitative risk management applications, including finance and insurance, among many others. EVT informs the selection of models which allow to perform inference at scales that lie outside the range covered by observed data. Underlying EVT there are a series of assumptions (e.g. membership in a domain of attraction) which are challenging (sometimes impossible) to verify in practice. And, even if the assumptions hold, the validity of the inference is only asymptotically correct, as the sample size increases to infinity.

This talk discusses an approach aimed at robustifying the inference in the context of EVT. In particular, given a performance measure of interest, say the evaluation of a quantile, we propose computing the worst-case value of such performance measure among all models which differ from a given baseline model (informed by EVT) by some tolerance. The key words are: all, differ, and tolerance. We present a calculus of variations approach (defining metrics between probability distributions) which is tractable (i.e., the solution can actually be evaluated in closed form), and allows to systematically address the issue of robust inference (meaning, the inference accounts for the possibility of violating the assumptions of EVT and the fact that the sample size is finite). Our methodology connects EVT with areas such as distributionally robust optimization and optimal transport.

References

[1] J. Blanchet and K. Murthy. *Distributionally robust tail modeling.* In preparation.

Generalized additive models for conditional dependence structures

Valérie Chavez-Demoulin
(joint work with Thibault Vatter)

We develop a generalized additive modeling framework for taking into account the effect of predictors on the dependence structure between two variables. We consider dependence or concordance measures that are solely functions of the copula, because they contain no marginal information: rank correlation coefficients or tail-dependence coefficients represent natural choices. We propose a maximum penalized log-likelihood estimator and derive its root-n-consistency and asymptotic normality. Finally, we present the results from a simulation study and apply the new methodology to a real dataset.

References

[1] T. Vatter and V. Chavez-Demoulin (2015). *Generalized additive models for conditional dependence structures.* Journal of Multivariate Analysis **141**, 147–167.

Black Swans, endogenous risk and price-mediated contagion

Rama Cont
(joint work with Lakshithe Wagalath)

The classical approach in quantitative risk management is to model risk as arising from an exogenous random process, representing “market randomness”. The underlying justification for this approach is that market fluctuations resulting from the supply and demand of numerous “small” market participants acting in an uncoordinated manner lead, through a central-limit type argyment, to a statistical description of fluctuations of price, volume and other aggregate quantities. Yet, such statistical approaches, even after accounting for the heavy tails encountered in financial data, fail to account for the huge financial losses triggered by “market events” such as the ones encountered in recent and not-so-recent market crises such as the October 1987 equity crash, the Quant Crash of August 2007 or the 2008 crisis. During such episodes, the assumptions underlying such Central Limit-type arguments break down, either through the destabilizing impact of one or more ”large” market participants or because many market participants trade in the same direction, due to similar portfolio constraints, leading to a lack of ’independence’ across components of the system.

These episodes have been labeled “Black Swans” or “perfect storms”, implying a lack of predictabilty and an analogy with natural catastrophes. We argue, on the contrary, that these –and many other– market dislocations, are in fact manifestations of **endogenous risk** arising from a destabilizing feedback mechanism in which temporary but systematic supply-demand imbalances and price moves affect each other to generate large price moves and spikes in volatility and correlation, even in absence of any large exogenous shock [1].

We show that such feedback loops can be modeled in a simple and parsimonious manner by integrating the *market impact* of supply-demand imbalance into any standard stochastic model of price changes [2]. We study examples of such models in a multi-asset setting and show they may be used to model the risk triggered by liquidation events, as well as temporary spikes in volatility and correlation which are observed in market crises. These findings have implications for the risk management of large financial institutions: we propose simple add-ons to current quantitative risk management models which such instituions may use to quantify the "endogenous risk" they may generate through their own actions [3].

References

[1] R. Cont and L. Wagalath (2013). *Running for the exit: distressed selling and endogenous correlation in financial markets*, Mathematical Finance, **23**(4), 718–741.

[2] R. Cont, L. Wagalath (2012). *Fire sale forensics: measuring endogenous risk.* To appear in Mathematical Finance, http://ssrn.com/abstract=2051013.

[3] R. Cont and L. Wagalath (2015). *Risk management for whales: dealing with market impact and endogenous risk.* Working paper.

Data mining for extreme behavior with application to ground level ozone

Dan Cooley

(joint work with Brook T. Russell)

Ground level ozone is a harmful pollutant that negatively affects people as well as plant species, and these negative effects are intensified when ozone is at its most extreme levels. This project aims to increase understanding of the meteorological conditions which lead to extreme ground level ozone conditions. We are motivated by the problem that atmospheric chemistry models are able to predict high ozone levels, but do not predict extreme ozone levels well.

Our approach focuses only on the tail behavior by utilizing the framework of regular variation. Our approach has two parts. The first is an optimization problem: given a set of meteorological covariates, we aim to find the linear combination which has the highest degree of tail dependence with ozone. The second is a data mining problem: given a long list of possible meteorological covariates, we seek to find the ones which are linked to extreme ozone.

We use a constrained optimization procedure which maximizes a measure of tail dependence and whose constraint enforces a requirement on the marginal distribution. Our optimization procedure requires that we consider tail dependence estimators with a smooth threshold, rather than the hard threshold typical of extremes, and we show consistency of estimators with smoothed thresholds. Data mining is performed within the model selection context, and because the model space cannot be explored completely, we employ an automated model search procedure. We present a simulation study which shows that the method can detect complicated conditions leading to extreme responses, and further that our approach protects us from overfitting.

We apply the method to ozone data for Atlanta and Charlotte and find similar meteorological drivers for these two Southeastern US cities. While several of these meteorological covariates are known to be linked to ozone concentrations, our procedure suggests some additional covariates which may influence extreme ozone levels. We identify some covariates which are likely linked to local causes and identify others which are common to the two cities.

References

[1] B.T. Russell, D. Cooley, W.C. Porter, B.J. Reich and C.L. Heald (2015). *Data mining for extreme behavior with application to ground level ozone. arXiv:1504.08080.*

Observation-driven time-varying parameter models and the generalized autoregressive method of moments

Drew Creal
(joint work with Siem Jan Koopman, André Lucas and Marcin Zamojski)

We gave an overview of our work on Generalized Autoregressive Score processes that allow time-variation in the parameters of a fully parametric time series model. Then, we introduce a new estimation framework that extends the Generalized Method of Moments (GMM) to settings where a subset of the parameters vary over time with unknown dynamics. To filter out the dynamic path of the time-varying parameter, we approximate the dynamics by an autoregressive process driven by the score of the local GMM criterion function. Our approach is completely observation driven, rendering estimation and inference straightforward. It provides a unified framework for modeling parameter instability in a context where the model and its parameters are only specified through (conditional) moment conditions, thus generalizing approaches built on fully specified parametric models. We provide examples of increasing complexity to highlight the advantages of our method.

References

[1] D. Creal, S.J. Koopman and A. Lucas (2013). *Generalized autoregressive score models with applications.* Journal of Applied Econometrics **28**(5), 777–795.
[2] D. Creal, S.J. Koopman and A. Lucas (2011). *A dynamic multivariate heavy-tailed model for time-varying volatilities and correlations.* Journal of Business & Economic Statistics **29**(4), 552–563.
[3] D. Creal, S.J. Koopman, A. Lucas and M. Zamojski (2015). *Generalized autoregressive method of moments.* University of Chicago Booth School of Business, working paper.

Statistics of extremes: challenges and opportunities

Miguel de Carvalho

In this talk I provide a personal view on some recent concepts and methods of statistics of extremes, and I discuss challenges and opportunities which could lead to potential future developments. Measure-dependent measures are here discussed as a natural probabilistic concept for modeling bivariate extreme values, and predictor-dependent spectral measures are discussed as a natural concept for modeling extremal dependence structures which vary according to a covariate. Families of g-tilted measures are introduced as a unifying device connecting some recently proposed approaches. En passant, I discuss a new estimator for the so-called scedasis function.

The main concepts of interest are defined below.

Probabilistic structures of interest

Definition 1. *Let $\mathcal{F}$ be the space of all probability measures that can be defined over $(\Omega_0, \mathcal{A}_0)$. If G_H is a probability measure on $(\Omega_1, \mathcal{A}_1)$, for all $H \in \mathcal{H} \subseteq \mathcal{F}$, then we say that G_H is a measure-dependent measure. The family $\{G_H : H \in \mathcal{H}\}$ is said to be a set of measure-dependent measures, if G_H is a measure-dependent measure.*

Below, let $\mathscr{H}$ denote the space of all probability measures H which can be defined over $([0,1], \mathbb{B}_{[0,1]})$, where $\mathbb{B}_{[0,1]}$ is the Borel sigma-algebra on $[0,1]$, and which obey the mean constraint $\int_{[0,1]} wH(\mathrm{d}w) = 1/2$.

Definition 2. *The family $\{H_x : x \in \mathcal{X}\}$ is a set of predictor-dependent spectral measures if $H_x \in \mathscr{H}$, for all $x \in \mathcal{X}$.*

Definition 3. *Let $\mathcal{F}$ be the space of all probability measures that can be defined on $(\Omega, \mathcal{A})$. Let $g_{i,I} : \Omega \mapsto \mathbb{R}$, for $i = 1, \dots, I$. A family of probability measures in $\mathcal{F}$, $\{F_1, \dots, F_I\}$, is a g-tilted family if there exists $F_0 \in \mathcal{F}$ and a functional θ such that*

$$\Phi_\theta(y) := \left(\frac{\theta(F_i)}{\theta(F_0)}\right)(y) = g_{i,I}(y), \quad y \in \Omega.$$

References

[1] M. de Carvalho. *Statistics of extremes: challenges and opportunities* in *Handbook of Extreme Value Theory and its Applications to Finance and Insurance.* Edited by F.M. Longin. Wiley (to appear 2015–2016).

[2] M. de Carvalho and A.C. Davison (2014). *Spectral density ratio models for multivariate extremes.* Journal of the American Statistical Association **109**, 764–776.

[3] M. de Carvalho, B. Oumow, J. Segers and M. Warcho (2013). *A Euclidean likelihood estimator for bivariate tail dependence.* Communications in Statistics—Theory and Methods **42**, 1176–1192.

[4] J. H. J. Einmahl, L. de Haan and C. Zhou (2015, in press). *Statistics of heteroscedastic extremes.* Journal of the Royal Statistical Society, Series B (DOI: 10.1111/rssb.12099).

[5] W. Fithian and S. Wager (2015). *Semiparametric exponential families for heavy-tailed data.* Biometrika **102**, 486–493.

Non-Gaussian bridge sampling with applications

Jin-Chuan Duan
(joint work with Changhao Zhang)

This paper provides a new bridge sampler that can efficiently generate sample paths, subject to some endpoint condition, for non-Gaussian dynamic models. This bridge sampler uses a companion pseudo-Gaussian bridge as the proposal and sequentially re-simulates sample paths via a sequence of tempered importance weights in a way bearing resemblance to the density-tempered sequential Monte Carlo method used in the Bayesian statistics literature such as in [1] and [2]. This bridge sampler is further accelerated by employing a novel idea of k-fold duplicating a base set of sample paths followed by support boosting. We implement this bridge sampler on a GARCH model estimated to the S&P 500 index series, and our implementation covers both parametric and non-parametric conditional distributions. Our performance study reveals that this new bridge sampler is far superior to either the simple-rejection method when it is applicable or other alternative samplers designed for paths with a fixed endpoint. Two applications are demonstrated – computing SRISK of the NYU-Stern Volatility Institute and infill maximum likelihood estimation.

References

[1] P. Del Moral, A. Doucet and A. Jasra (2006). *Sequential Monte Carlo samplers.* Journal of the Royal Statistical Society (Series B), **68**(3), 411–436.
[2] J.-C. Duan and A. Fulop (2015). *Density-tempered marginalized sequential Monte Carlo samplers.* Journal of Business and Economic Statistics **33**(2), 192–202.

Long run risk management: scenario generation for the term structure

Robert Engle
(joint work with Emil Siriwardane)

In the low volatility environment preceding the financial crisis, many firms increased their leverage and risk. Arguably investments in illiquid positions should have been evaluated with respect to long horizon volatility and risk. Risk managers should account for the risk that the risk can change. Scenario analysis is a solution to the need. A probability based scenario generator is developed to examine the long run risk of the US treasury term structure. It features a reduced rank vector autoregression with Nelson Siegel factors and GARCH-DCC multivariate disturbances. Backtests motivate model improvements.

References

[1] J. Christensen, J. Lopez and G. Rudebusch (2013). *A probability based stress test of federal reserve assets and income.* Federal Reserve Bank of San Francisco, working paper.
[2] F. Diebold and G. Rudebusch (2013). *Yield Curve Modeling and Forecasting: The Dynamic Nelson Siegel Approach.* Princeton University Press.

[3] A. Monfort, F. Pegararo, J.-P. Renne and Guillaume Roussellet (2014). *Staying at zero with affine processes: a new dynamic term structure model.* Banque de France, working paper.
[4] U. Müller, M. Dacorogna and P. Blum (2010). *Bootstrapping the economy – a non-parametric method of generating consistent future scenarios.* SCOR, working paper.

Replicating portfolio approach to solvency capital calculation

Damir Filipović
(joint work with Mathieu Cambou)

The calculation of solvency capital for life insurance portfolios is a challenging task, which has has not gained much attention in the literature yet.

The new European regulatory framework Solvency II and the Swiss Solvency Test require the modeling of the profit and loss distribution of the asset-liability portfolio on a one-year time horizon, see [1] and [2]. Solvency capital in the Swiss Solvency Test is then determined as 99%-expected shortfall of this profit and loss (99.5%-value at risk in Solvency II), see [3] for a definition of expected shortfall and value at risk.

The value of the liabilities is defined as expected deflated liability cash flows. For life insurance this requires simulations of cash flows up to 40 years and beyond. These simulations are computationally costly. The modeling of the value of the liabilities in one year, say L, cannot be done via nested simulations therefore. The replicating portfolio approach consists in projecting the terminal, say time T, value of discounted liability cash flows, Z, onto the subspace in L^2 generated by a family of m replicating assets, whose discounted time-1 values are denoted by $\mathbf{A}$. This yields a replicating portfolio with time-1 value given by $\phi^\top \mathbf{A}$, for the ϕ from the L^2 projection, which we use as proxy for L. We show that, as m tends to infinity, the expected shortfall of $\phi^\top \mathbf{A}$ converges to the expected shortfall of L. This proves that the replicating portfolio approach to solvency capital calculation works asymptotically.

We then study the estimation error of this approximation that results from the finite sampling of Z . We show that the simulation based estimator of ϕ is unbiased and satisfies a central limit theorem as the number n of simulations tends to infinity.

The total estimation error of the expected shortfall of L that results from approximating L by the replicating portfolio $\phi^\top \mathbf{A}$ with m instruments and the simulation of n samples is shown to be decreasing in n. However, the estimation error from simulation is increasing in m. As a consequence, for any simulation time budget n there is an optimal number of replicating instruments m, to be determined case by case. Some numerical examples illustrate our findings.

We also discuss some open issues such as the optimal choice of replicating instruments and variance reduction of the simulation based esimator of ϕ, see [4].

References

[1] European Insurance and Occupation Pensions Authority EIOPA. *QIS5 Technical Specifications.* Brussels, July 2010.

[2] Federal Office of Private Insurance FOPI. *Technical Document on the Swiss Solvency Test.* October 2006.

[3] H. Föllmer and A. Schied (2012). *Stochastic Finance.* Walter de Gruyter, Berlin, New York.

[4] P. Glasserman (2004). *Monte Carlo Methods in Financial Engineering.* Springer.

CRIX or evaluating Blockchain based currencies

Wolfgang K. Härdle

(joint work with Simon Trimborn)

More and more companies start offering digital payment systems. Smartphones evolve to a digital wallet such that it seems like we are about to enter the era of digital finance. In fact we are already inside a digital economy. The market of e-x ($x =$ "finance", "money", "book", you name it ...) has not only picked up enormous momentum but has become standard for driving innovative activities in the global economy. A few clicks at y and payment at z brings our purchase to location w. Own currencies for the digital market were therefore just a matter of time. The idea of the Nobel Laureate Hayek, see [1], to let companies offer concurrent currencies seemed for a long time scarcely probable, but the invention of the Blockchain made it possible to fill his vision with life. Cryptocurrencies (abbr. cryptos) surfaced and opened up an angle towards this new level of economic interaction. Since the appearance of bitcoins, several new cryptos spread through the web and offered new ways of proliferation. The crypto market then fanned out and showed clear signs of acceptance and deepening liquidity so that a closer look at the general moves and dynamics is called for. CRIX - a CRyptocurrency IndeX, `http://crix.hu-berlin.de`, has been created for this purpose, Figure 1. CRIX follows the so-called Laspeyres' index idea:

$$\text{CRIX}(k)_t = \frac{\sum_i^k MV_{it} \cdot AW_{it}}{Divisor} \tag{1}$$

where MV_{it} is the market capitalization of the crypto i at time point t and k the number of constituents. AW_{it} is the adjusted weight, defined as

$$AW_{it} = \frac{CW_{it}}{W_{it}} \tag{2}$$

with CW_{it} the capped weight and $W_{it} = \frac{MV_{it}}{\sum_i MV_{it}}$ the weight the crypto i would normally have in CRIX. The weight will be capped if a single crypto i would have an influence of 50% or more in CRIX. The cap is part of the index rules since the analysis of the trading volume showed that bitcoin has a major influence in the market even though its trading volume, relative to all outstanding bitcoins, is much lower than for alternative cryptocurrencies. This implies a higher interest of interested parties in alternative cryptos than their market value suggests, which motivates to lower the influence of bitcoin. The *Divisor* with the starting value

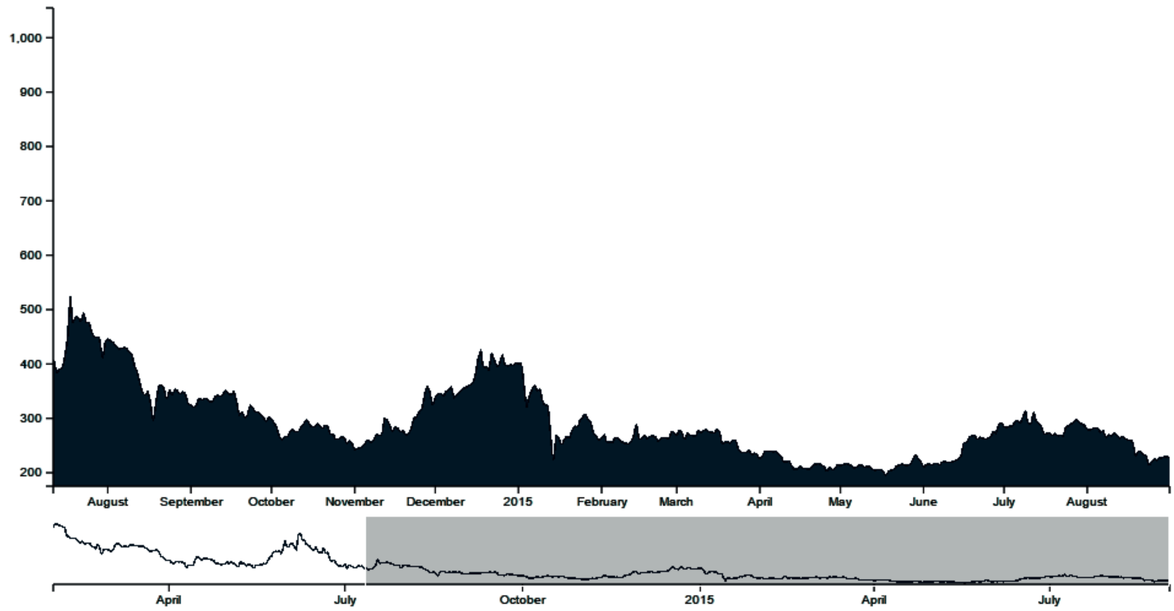

FIGURE 1. Snapshot of the CRIX website on the 24.09.2015

$\frac{\sum_i MV_i}{1000}$ controls that CRIX is not affected by a shifting of its constituents, just by price changes. To ensure this, it is adjusted when necessary:

$$\frac{\sum_i MV_{i,t-1}}{Divisor_{t-1}} = \text{CRIX}_{t-1} = \text{CRIX}_t = \frac{\sum_j MV_{j,t}}{Divisor_t}. \tag{3}$$

The index rules, which form the CRIX methodology, ensure that CRIX reacts fast and dynamically to changes in the market, such that it gives insight into the evolvement of cryptos which surfaced in the digital economy. CRIX relies on liquidity measures and on the Bayesian Information Criterion (BIC), see [3]. BIC is used to decide how many cryptos shall participate in a representative proxy of the market. CRIX will be the perfect benchmark, if the amount of constituents is always optimal. For this purpose, a procedure was created which compares the difference between the total market (all market participants) and several candidate indices,

$$\varepsilon_{j,t} = \text{total market}_t - \text{CRIX}(k)_{j,t}, \tag{4}$$

where $\text{CRIX}(k)_{j,t}$ is the CRIX version j with k_j constituents and $\varepsilon_{j,t}$ is the respective difference. The total market is represented by an index of all market participants, which is computed by the formulas (1), (2) and (3). The candidate indices, $\text{CRIX}(k)_j$, have different amounts of constituents which fulfill $k_1 < k_2 < k_3 < \cdots$. The BIC criterion evaluates the differences, $\varepsilon_{j,t}$, between the candidates and the total market with the respective likelihood L_j,

$$L_j = \prod_t f_j(\varepsilon_{j,t}), \tag{5}$$

where f_j represents the density of the $\varepsilon_{j,t}$ over all t. It penalizes L_j with the amount of constituents, k_j, such that the following formula results:

$$BIC_j = -2 \log L_j + k_j \cdot \log(n_j), \tag{6}$$

where n_j is the number of observations. The density, f_j, is estimated nonparametrically with a Gaussian kernel. Since the same data are used to estimate f_j and the BIC_j, a "leave-one-out" cross-validation procedure is performed in order to overcome the bias; see [4]. The search for the optimal model terminates at level j whenever

$$BIC_{j-1} < BIC_j. \tag{7}$$

The entire procedure runs every third month and the resulting number of index members, k, will be fixed for the coming 3 months. A detailed version of the methodology can be found on the website, `http://crix.hu-berlin.de`.

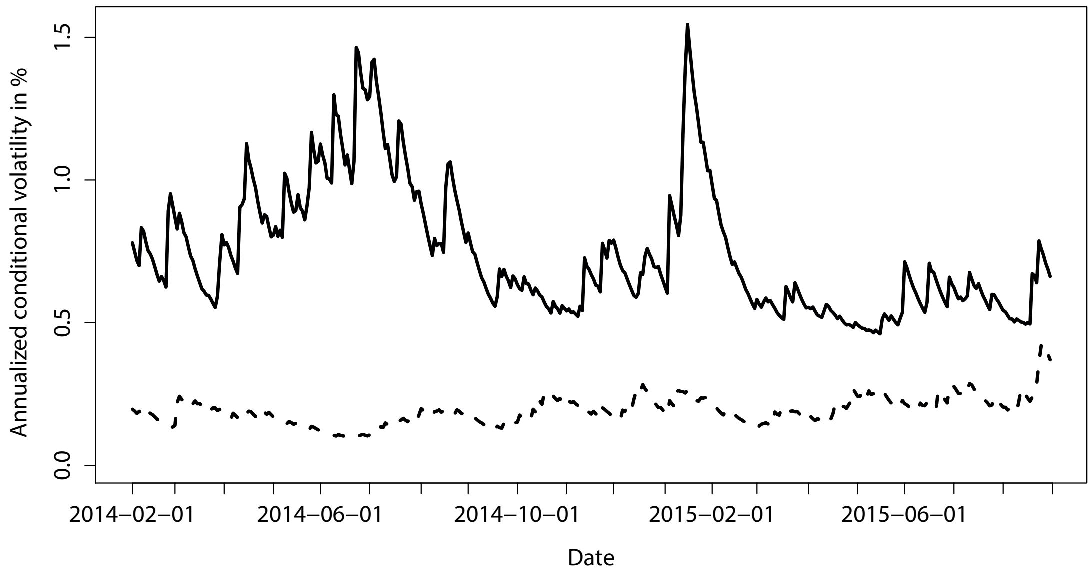

FIGURE 2. Annualized conditional volatility in % of ▬ CRIX and - - - DAX

"Satoshi Nakamoto" described in "his" paper, see [2], a decentralized payment system. While many just think about bitcoin and other cryptos as currencies, some argue that cryptos can be seen as commodities, see e.g. [6]. Being treated as a commodity, makes it a store of value and by this means an exchangeable, investable product. CRIX was created to investigate this feature of cryptos by comparing the crypto market against other investment universes and classifying CRIX in terms of economic risk against them. We perform our analysis on data in the time period 01.02.2014 - 01.09.2015 and observe, that the annualized conditional volatility, measured with a GARCH(1,1) model, has a higher base level than the

DAX but that the amount of high spikes in the volatility decreases, see Figure 2. This indicates us, that CRIX bears a higher risk than the german bluechip index but is stabilizing although on a higher level. The detected Expected Shortfall (ES) lies far away from that of fiat fx rates, where the ES is defined as the conditional expectation

$$\mathrm{E}\left[X|X < x_{0.01}\right] \tag{8}$$

with $x_{0.01}$ the 0.01-quantile and assuming the tails to follow a generalized Pareto distribution, see [5]. The risk level, which ES indicates, lies much closer to risky stock markets like the Greece or Russian one, see Table 1.

	ES
CRIX	-0.1579
SP500	-0.0528
DAX	-0.0700
RTSI	-0.1343
ATHEX	-0.1288
EUR to USD	-0.0223
RUB to USD	-0.0637

Table 1. Extreme value theory based ES at $\alpha = 0.01$ and with threshold $u_t = 0.1$ for CRIX, SP500, DAX, RTSI, ATHEX, EUR to USD and RUB to USD.

Finally, option prices are computed to attach a price tag to the risk which CRIX bears. Based on these insights one may conclude that if options would exist for CRIX, they would currently be so expensive that it - most likely - doesn't pay out to protect ones investment with them. Besides these early findings, it appears that this market is stabilizing and qualifies itself little by little as a serious investment alternative. CRIX and the risk statistics will be computed continuously and be published on `http://crix.hu-berlin.de` to offer interested parties a comprehensive overview.

References

[1] F.A. Hayek (1976). *Denationalization of Money: An Analysis of the Theory and Practice of Concurrent Currencies.* Institute of Economic Affairs, London.
[2] S. Nakamoto (2008). *Bitcoin: a peer-to-peer electronic cash system.*
[3] G. Schwarz(1978). *Estimating the dimension of a model.* Annals of Statistics **6**, 461–464.
[4] A. Potapov, J.R. Muirhead, S.R. Lele and M.A. Lewis (2011). *Stochastic gravity models for modeling lake invasions.* Ecological Modelling **222**, 964–972.
[5] A.J. McNeil, R. Frey (2000). *Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach.* Journal of Empirical Finance **7**, 271–300.
[6] W. Buiter (2014). *Gold: a six thousand year-old bubble revisited.* Citi Research Economics.

Estimating the spot covariation of asset prices – statistical theory and empirical evidence

Nikolaus Hautsch
(joint work with Markus Bibinger, Peter Malec and Markus Reiss)

We propose a new estimator for the spot covariance matrix of a multi-dimensional continuous semi-martingale log asset price process which is subject to noise and non-synchronous observations. Let $(X_t)_{t\geq 0}$ denote the d-dimensional efficient log-price process. We assume that X_t follows a continuous Itô semi-martingale

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dB_s,\; t\in[0,1], \tag{1}$$

defined on a filtered probability space $\left(\Omega,\mathcal{F},(\mathcal{F})_{t\geq 0},\mathbb{P}\right)$ with drift b_s, d-dimensional standard Brownian motion B_s and instantaneous volatility matrix σ_s. The latter yields the $(d\times d)$-dimensional spot covariance matrix $\Sigma_s=\sigma_s\sigma_s^\top$, which is our object of interest. We consider a setting in which discrete and non-synchronous observations of the process (1) are diluted by market microstructure noise, i.e.,

$$Y_i^{(p)} = X^{(p)}_{t_i^{(p)}} + \epsilon_i^{(p)}\,,\; i=0,\ldots,n_p,\; p=1,\ldots,d\,, \tag{2}$$

with observation times $t_i^{(p)}$, and observation errors $\epsilon_i^{(p)}$.

The key idea is to approximate the underlying process (1) in model (2) by a process with block-wise constant covariance matrices and noise levels. Using spectral statistics, we construct an unbiased estimator for the block-wise covariance estimates. Computing an optimally weighted average of these block-wise statistics across all spectral frequencies yields an efficient block-wise covariance estimate. The optimal weights are given proportionally to the local Fisher information matrices. This gives rise to a local method of moments (LMM) estimator as proposed by [1]. The final spot estimator is then constructed by smoothing over adjacent blocks.

We prove consistency and a point-wise stable central limit theorem for the proposed spot covariance estimator in a general setup with stochastic volatilities, leverage and for general noise distributions. We show that the approach attains rate-optimality. Moreover, we allow the market microstructure noise being autocorrelated and propose a method to adaptively infer the autocorrelations from the data. Based on simulations we provide empirical guidance on the implementation of the estimator and show how to optimally choose the length of blocks and the smoothing interval, and the spectral cut-off point. Moreover, we demonstrate that the proposed procedure for estimating the order of serial dependence in the microstructure noise process performs well under realistic conditions.

We apply the proposed estimator to high-frequency data of a cross-section of NASDAQ blue chip stocks. Estimating spot covariances, correlations and volatilities in normal but also unusual periods yields novel insights into intraday covariance and correlation dynamics. We show that intraday (co-)variations (i) follow underlying periodicity patterns, (ii) reveal substantial intraday variability associated with (co-)variation risk, (iii) are strongly serially correlated, and (iv) can increase strongly and nearly instantaneously if new information arrives.

References

[1] M. Bibinger, N. Hautsch, P. Malec and M. Reiss (2014). *Estimating the quadratic covariation matrix from noisy observations: local method of moments and efficiency.* Annals of Statistics **42**, 1312–1346.

Improved algorithms for computing worst Value-at-Risk: numerical challenges and the adaptive rearrangement algorithm

Marius Hofert
(joint work with Amir Memartoluie, David Saunders and Tony Wirjanto)

In risk aggregation, a topic in the realm of Quantitative Risk Management, a currently discussed problem is how to compute the Value-at-Risk $\mathrm{VaR}_\alpha(L^+)$ of the sum $L^+ = \sum_{j=1}^d L_j$ of risks $L_1 \sim F_1, \dots, L_d \sim F_d$. The marginal distribution functions $F_1, \dots, F_d$ are typically known (or can be estimated), whereas there is considerable uncertainty concerning the dependence structure of $(L_1, \dots, L_d)$. When treating the latter as unknown, the goal is to compute lower and upper bounds on $\mathrm{VaR}_\alpha(L^+)$ among all distributions with margins $F_1, \dots, F_d$, i.e., $\underline{\mathrm{VaR}}_\alpha(L^+)$ and $\overline{\mathrm{VaR}}_\alpha(L^+)$, respectively; we focus on the latter, which is also termed *worst Value-at-Risk*.

In the homogeneous case, i.e., $F := F_1 = \dots = F_d$, there are two approaches known for computing $\overline{\mathrm{VaR}}_\alpha(L^+)$. In the present work, we highlight and solve several numerical issues when implementing these (supposedly "explicit") approaches in statistical software. In particular, we construct a proper initial interval for the root-finding procedure involved in computing $\overline{\mathrm{VaR}}_\alpha(L^+)$ with the approach of [1, Prop. 1] for F being Pareto (i.e., $F(x) = 1 - (1 + x)^{-\theta}$).

In the inhomogeneous case, we consider the Rearrangement Algorithm of [2], [3]. We study the algorithm under various scenarios and conclude with three improvements of the algorithm. First, we build in relative (instead of absolute) tolerances; this helps in choosing more reasonable default tolerances independently of the actual computed $\overline{\mathrm{VaR}}_\alpha(L^+)$. Second, we introduce a *joint* relative tolerance; this guarantees that the computed bounds for $\overline{\mathrm{VaR}}_\alpha(L^+)$ are sufficiently close. And finally, we adaptively choose the number N of discretization points; this takes away the need to choose a reasonable N (which actually also depends on the required joint relative tolerance). The resulting algorithm is termed *Adaptive Rearrangement Algorithm* and available in the R package `qrmtools` (besides the approaches mentioned above).

Overall, we very often realize that there are significant hurdles to overcome in order to apply theoretical results in practice. In particular, this includes the actual implementation in (statistical) software, which has not been addressed before in the realm of Quantitative Risk Management. We started to term this new area of research *Computational Risk Management*.

References

[1] P. Embrechts, G. Puccetti, L. Rüschendorf, R. Wang and A. Beleraj (2014). *An academic response to Basel 3.5.* Risks **2**(1), 25–48.

[2] G. Puccetti and L. Rüschendorf (2012). *Computation of sharp bounds on the distribution of a function of dependent risks.* Journal of Computational and Applied Mathematics **236**(7), 1833–1840.

[3] P. Embrechts, G. Puccetti and L. Rüschendorf (2013). *Model uncertainty and VaR aggregation.* Journal of Banking and Finance **37**(8), 2750–2764.

Two examples of efficient importance sampling in finance

Henrik Hult
(joint work with B. Djehiche, P. Nyquist and J. Nykvist)

In this talk I will present the connection between subsolutions of first-order Hamilton-Jacobi equations and design of efficient rare-event simulation algorithms. In particular it will shown how subsolutions corresponding to efficient algorithm can be constructed from the Mañé potential. This approach to the construction of subsolutions also lead to a minmax representation of viscosity solutions to Hamilton-Jacobi equation. I will demonstrate an application to computing risk probabilities for a portfolio of possibly path-dependent derivatives as well as an application to computing the credit value adjustment for a portfolio of interest rate derivatives.

References

[1] B. Djehiche, H. Hult and P. Nyquist (2015). *Minmax representations of viscosity solutions of Hamilton-Jacobi equations and applications in rare-event simulation. arXiv:1406.3605.*

[2] H. Hult and J. Nykvist (2015). *Efficient importance sampling to compute loss probabilities in derivative portfolios.* In J. Nykvist, *Topics in importance sampling and derivatives pricing.* KTH, PhD thesis, `http://kth.diva-portal.org/smash/get/diva2:786454/FULLTEXT01.pdf`.

Refined extremal dependence of stochastic volatility models

Anja Janssen
(joint work with Holger Drees)

There exists a large variety of models for financial time series and the different classes of models vary in their extremal behavior. Some of these models for a stationary time series $(X_t)_{t\in\mathbb{Z}}$ exhibit so-called asymptotic dependence, which means that

$$\lim_{x\to\infty} P(|X_{t+h}| > x \mid |X_t| > x) > 0, \tag{1}$$

for some lags $h \neq 0$. The popular class of GARCH(p, q) models is a prominent example for this. Other models show asymptotic independence, i.e. the limits in (1) are equal to 0 for all lags $h \neq 0$. This holds in particular true for many specifications of stochastic volatility models. It is, however, difficult to decide which class is most appropriate for a given application because in reality we are never able to witness the behavior of exceedances over *infinitely large* thresholds and models from both classes may show clustering of the largest *observed* values.

We are aiming at a refined analysis of extremal dependence for time series by investigating the joint extremal behavior of lagged observations (X_t, X_{t+h}) in the framework of regular variation on cones (cf., for example, [3]). We show that many of the common model specifications for stochastic volatility models exhibit a very strong form of asymptotic independence (meaning that the coefficient of tail dependence, cf. [2], equals 1/2 for all lags which is the same as for i.i.d. observations). This motivates us to develop a new class of stochastic volatility models with a heavy-tailed volatility sequence and light-tailed innovations which allow for a much more flexible second order structure, cf. [1]. As an auxiliary result of our analysis we show an extension of Breiman's lemma to regular variation on the cone $(0, \infty)^d$.

References

[1] A. Janßen and H. Drees (2015). *A stochastic volatility model with flexible extremal dependence structure.* Bernoulli, forthcoming.

[2] A.W. Ledford and J.A. Tawn (1996). *Statistics for near independence in multivariate extreme values.* Biometrika **83**, 169–187.

[3] S.I. Resnick (2008). *Multivariate regular variation on cones: application to extreme values, hidden regular variation and conditioned limit laws.* Stochastics **80**, 269–298.

Spectral functions of stationary max-stable processes

Zakhar Kabluchko

(joint work with Clément Dombry)

Consider a stationary, stochastically continuous, max-stable process $(\eta(t))_{t\in T}$ on $T = \mathbb{R}^d$. Suppose for concreteness that η has unit Fréchet margins. A fundamental theorem of L. de Haan [1] states that η can be represented as

$$(\eta(t))_{t\in T} \stackrel{f.d.d}{=} \left(\max_{i\in\mathbb{N}} U_i Y_i(t)\right)_{t\in T},$$

where $(U_i)_{i\in\mathbb{N}}$ is an enumeration of the points of a Poisson point process on $(0, +\infty)$ with intensity measure $u^{-2}\mathrm{d}u$, and, independently, $(Y_i)_{i\in\mathbb{N}}$ are i.i.d. copies of a non-negative process $(Y(t))_{t\in T}$ such that $\mathbb{E}[Y(t)] = 1$. In the present talk, we state a number of results relating ergodic properties of η to the properties of its "spectral function" Y. For example, the process η is ergodic if and only if $\liminf_{R\to\infty} \frac{1}{R^d} \sum_{|t|\le R} Y(t) = 0$ almost surely. The process η is mixing if and only if $\lim_{|t|\to\infty} Y(t) = 0$ in probability. In the case when the sample paths of η are locally bounded, the process η has a mixed moving maximum representation if and only if $\lim_{|t|\to\infty} Y(t) = 0$ almost surely. Similar results (with integrals replaced by sums) can be obtained for processes on $T = \mathbb{Z}^d$.

References

[1] L. de Haan (1984). *A spectral representation for max-stable processes.* Annals of Probability **12**(4), 1194–1204.
[2] C. Dombry and Z. Kabluchko (2015). *In preparation.*

Max-linear models on directed acyclic graphs

Claudia Klüppelberg
(joint work with Nadine Gissibl)

We consider a new structural equation model, where all random variables can be written as a max-linear function of their parents and independent noise terms.We assume that the dependence structure can be modeled by a directed acyclic graph. We show that the resulting multivariate distribution is max-linear and characterize all max-linear models, which are generated by a structural equation model. We detail the relation between the coefficients of the structural equation model and the max-linear coefficients. This leads to the presentation of a max-linear structural equation model as the solution of a fixed point equation, and to a unique minimal DAG characterising the model.

References

[1] N. Gissibl and C. Klüppelberg (2015). *Max-linear models on directed acyclic graphs.* In preparation.
[2] N. Gissibl and C. Klüppelberg (2015). *Regular conditional distributions of max-linear structural equation models.* In preparation.

Transparent valuation and aggregation of risks

Filip Lindskog
(joint work with Jonas Alm)

I will discuss an approach to valuation of liability cash flows and the consequences for aggregation of risks. The aim is to better assess the market-consistent value one year from now of a sum of dependent liability cash flows. The approach allows for separating and quantifying the effects on the future aggregate liability value of trends and variability in (claims-/price-) inflations, inflation-adjusted pure insurance risks, and nominal interest rate risk. Given actuarial forecasts of inflation-adjusted pure insurance risks, and a sensible model for nominal interest rates supported by interest rate data, what remains is just subjective modeling decisions regarding long terms trends in inflation and nominal interest rates. The approach presented here makes these modeling decisions visible and allows their effects to be easily quantified and compared to the other effects. Model selection and validation issues will be discussed. The presentation is based partly on material in [1].

References

[1] J. Alm and F. Lindskog (2015). *Valuation of index-linked cash flows in a Heath-Jarrow-Morton framework.* Risks **3**, 338–364.

Extremes of Gaussian and its related processes

Jingchen Liu
(joint work with Xiaoou Li and Gongjun Xu)

Gaussian processes are employed to model spatially varying errors in various stochastic systems. In this talk, we consider the analysis of the extreme behaviors for such systems. In particular, the topic covers various nonlinear functionals of Gaussian processes including the supremum norm, integral of convex functions, and stochastic partial differential equations with random coefficients. We present asymptotic results for the associated rare-event probabilities.

References

[1] Xiaoou Li, Jingchen Liu and Gongjun Xu. *On the tail asymptotics of exponential integrals of Gaussian random fields with small noise.* To appear in Mathematics of Operations Research.
[2] Jingchen Liu (2012). *Tail Approximations of integrals of Gaussian random fields.* Annals of Probability **40**(3), 1069–1104.
[3] Jingchen Liu and Gongjun Xu (2012). *Some asymptotic results of Gaussian random fields with varying mean functions and the associated processes.* Annals of Statistics **40**(1), 262–293.
[4] Jingchen Liu and Gongjun Xu (2013). *On the density functions of integrals of Gaussian random fields.* Advances in Applied Probability **45**(2), 398–424.
[5] Jingchen Liu and Gongjun Xu (2014). *On the conditional distributions and the efficient simulations of exponential integrals of Gaussian random fields.* Annals of Applied Probability **24**(4), 1691–1738.

Modelling and managing longevity risk

Stéphane Loisel
(joint work with Nicole El Karoui and Yahia Salhi)

In this talk, we first explain challenges regarding human longevity modelling and forecasting. In particular, we present the impact of characteristics of individuals like education and income levels, as well as geographical location and marital status. We then introduce an optimal stopping problem related to sequential testing of changing longevity patterns. We show that the optimal strategy is given by the so-called cusum strategy, when the Lorden criterion is used. We discuss practical issues related to an illustration on a real-world longevity example.

References

[1] P. Barrieu, H. Bensusan, N. El Karoui, C. Hillairet, S. Loisel, C. Ravanelli and Y. Salhi (2012). *Understanding, modeling and managing longevity risk: key issues and main challenges.* Scandinavian Actuarial Journal **3**, 203–231.

[2] N. El Karoui, S. Loisel and Y. Salhi (2015). *Minimax optimality in robust detection of a disorder time in Poisson rate.* Submitted.

[3] G.V. Moustakides (1986). *Optimal stopping times for detecting changes in distributions.* Annals of Statistics, **14**(4), 1379–1387.

Semi-parametric models for multivariate extreme value distributions

John Nolan

(joint work with Anne-Laure Fougères and Cécile Mercadier)

We present a new way to estimate multivariate extreme value distributions from data using max projections for different classes of semi-parametric models from [1]: discrete angular measure, generalized logistic, piecewise polynomial angular measures, and Dirichlet mixture models. The approach works in any dimension, though computation time increases quickly as dimension increases. The procedure requires tools from computational geometry and multivariate integration techniques. An `R` package `mvevd` is being developed to implement the method for the above classes, along with tools for simulating and calculating cumulative distribution functions.

References

[1] A.L. Fougéres, C. Mercadier and J.P. Nolan (2013). *Dense classes of multivariate extreme value distributions.* Journal of Multivariate Analysis **116**, 109–129.

On the law of the iterated logarithm for sequences of m-orthogonal random variables

Valentin V. Petrov

In Petrov [1] a theorem on the upper limit of a sequence of dependent random variables was proved. By means of this theorem some sufficient conditions were proved for the applicability of the law of the iterated logarithm to sequences of m-dependent random variables with finite variances. These results were used in Petrov [2] where the condition of m-dependence was replaced by the notion of m-orthogonality which was introduced in the same paper.

Let m be a non-negative integer. By definition, a sequence of random variables $\{X_n; n = 1, 2, \ldots\}$ on a probability space is a *sequence of m-orthogonal random variables* if $\mathbb{E}[X_n^2] < \infty$ for every n and $\mathbb{E}[X_k X_j] = 0$ if $|k - j| > m$. In particular, a sequence of 0-orthogonal random variables is a sequence of orthogonal random variables.

Many papers have been devoted to limit theorems for sequences of m-dependent random variables. Every sequence of m-dependent random variables with zero means and finite variances is a sequence of m-orthogonal random variables. This statement remains true if we replace the condition of m-dependence by the weaker condition of pairwise m-dependence.

Limit theorems for sequences of m-orthogonal random variables may represent some interest. The following theorem is a generalization of a result in Petrov [2].

Theorem 1. *Let $\{X_n\}$ be a sequence of m-orthogonal random variables with zero means. Put*

$$S_n = \sum_{k=1}^{n} X_k\,, \quad B_n = \mathbb{E}[S^n]\,, \quad a_n = (2\,B_n\;\log\log B_n)^{1/2}\,.$$

Suppose that $B_n \to \infty$, $B_n/B_{n+1} \to 1$ $(n \to \infty)$ and

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\max_{[c^n]\le k<[c^{n+1}]} S_k \ge (1+\varepsilon)\,a_{[c^n]}\Big) < \infty$$

for every $\varepsilon > 0$ and every $c > 1$. Then

$$\limsup_{n\to\infty} S_n/a_n \le 1 \qquad \text{a.s.}$$

References

[1] V.V. Petrov (1980). *On the law of the iterated logarithm for sequences of dependent random variables.* Zap. Nauch. Sem. LOMI. **97**, 186–194 (in Russian). English translation (1984): Journal of Soviet Mathematics **24**, 611–617.

[2] V.V. Petrov (1982), *Sequences of m-orthogonal random variables.* Zap. Nauch. Sem. LOMI **119**, 198–202 (in Russian). English translation (1984): Journal of Soviet Mathematics **27**, 3136–3140.

A test for the rank of the volatility process: the random perturbation approach

Mark Podolskij
(joint work with Jean Jacod)

In this talk we present a test for the maximal rank of the volatility process in the continuous diffusion framework. We consider a d-dimensional continuous diffusion process of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s, \qquad t \in [0,1],$$

where a is d-dimensional drift process and σ is a $d \times d$-dimensional volatility process. The underlying model is observed at high frequency, i.e. we are given the data points $X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{[1/\Delta_n]\Delta_n}$ with $\Delta_n \to 0$. Let r denote the rank process of the matrix $c = \sigma\sigma^\star$, i.e. $r_t = \text{rank}(c_t),\ t \in [0,1]$. Our aim is to estimate/test the maximal rank of c during a given trading day $[0,1]$. Hence, our object of interest is given via

$$R = \sup_{t\in[0,1)} r_t.$$

We remark that R is identical to the minimal number of independent Brownian motions required for modelling the process X during the period $[0,1]$. As such our statistical problem is directly related to the amount of factors in continuous factor models.

In order to approach the unknown random variable R we employ the following perturbation method. Let $A \in \mathbb{R}^{d\times d}$ be a given positive semidefinite matrix with $r = \text{rank}(A)$. We consider a positive definite matrix $B \in \mathbb{R}^{d\times d}$ and a number $h \searrow 0$. Our main observation is the asymptotic relationship

$$\det(A + hB) = h^{d-r}\gamma_{A,B} + O(h^{d-r+1})$$

with $\gamma_{A,B} := \sum_{C\in\mathcal{M}_{A,B}} \det(C)$ and

$$\mathcal{M}_{A,B} := \{C \in \mathbb{R}^{d\times d} :\ C_i = A_i \text{ or } C_i = B_i,\ \text{A and C share } r \text{ joint columns}\}.$$

When $\gamma_{A,B} \neq 0$, we deduce that

$$\frac{\det(A + 2hB)}{\det(A + hB)} \to 2^{d-r} \qquad \text{as} \quad h \searrow 0,$$

which gives a useful identification method of the unknown rank r. In the next step we apply this idea to the framework of continuous diffusion model. First, we introduce a random perturbation of the original diffusion process

$$Z_t^n = X_t + \sqrt{\Delta_n}\,\widehat{W}_t,$$

where $\widehat{W}$ is a new Brownian motion independent of previously defined processes. Our main test statistic $S(Z^n, \Delta_n)$ is defined via

$$S(Z^n, \Delta_n) = \sum_{i=1}^{[1/\Delta_n]-d+1} \det{}^2\left(\Delta_i^n Z^n/\sqrt{\Delta_n}, \ldots, \Delta_{i+d-1}^n Z^n/\sqrt{\Delta_n}\right)$$

with $\Delta_i^n Z^n = Z_{i\Delta_n}^n - Z_{(i-1)\Delta_n}^n$. When the drift a, the volatility σ and the volatility of volatility are following a continuous diffusion model, we can prove the convergence in probability $\Delta_n^{1+R-d} S(Z^n, \Delta_n) \to S$, where $S > 0$ almost surely. Hence, we immediately deduce the convergence

$$T_n := \frac{S(Z^n, 2\Delta_n)}{S(Z^n, \Delta_n)} \to 2^{d-R} \qquad \text{in probability,}$$

which obviously provides a consistent estimator of the unknown maximal rank R after a suitable transformation. In the final step we prove the associated central limit theorem, which enables us to derive a formal testing procedure for e.g. hypotheses of the type

$$H_0 :\ R = R_0 \qquad \text{vs.} \qquad H_1 :\ R \neq R_0,$$

where $R_0 \in \{0, \ldots, d\}$.

References

[1] J. Jacod and M. Podolskij (2013). *A test for the rank of the volatility process: the random perturbation approach.* Annals of Statistics **41**(5), 2391–2427.

Recent results in the mathematical modeling of financial bubbles

Philip Protter
(joint work with Shihao Yang and Yoshiki Obayashi)

Financial bubbles are often described as when the price of a risky asset (in this paper, we study stocks) has a market price that exceeds the price a rational person would pay for the stock, known as the *fundamental price*. The price a rational person would pay is typically considered to be the conditional expectation of the future cash flows of the stock, considered under a risk neutral measure. This definition is problematic, for while one can observe market prices, it is not really possible to calculate future cash flows.

This is where mathematics comes to the rescue. In a series of research papers it has become clear that on a finite time horizon the market price exceeds the fundamental price if and only if the market price is a strict local martingale under a selected risk neutral measure. Therefore to determine whether or not a given stock has bubble pricing or not, we need only to check whether or not the market price process is a strict local martingale, or a true martingale.

There is the issue of which risk neutral measure one should use? In simple cases that are nevertheless sophisticated enough to be useful, we can finesse this issue. For example, if we model the dynamic evolution of the stock price as a solution of a stochastic differential equation of the form

$$dX_t = \sigma(X_t)dB_t + b(X_t, Y_t)dt; \quad X_0 = 1 \tag{1}$$

where Y represents a stochastic process reflecting relevant market forces, then under any one of the infinite choice of risk neutral measures Q we have that the drift disappears via a Girsanov type transformation, to get

$$dX_t = \sigma(X_t)dB_t; \quad X_0 = 1 \tag{2}$$

and therefore it does not matter which risk neutral measure we use! Moreover in current work, we have been able to extend this idea to equations with stochastic volatility, although there are many caveats to this procedure:

$$dX_t = \sigma(X_t, \nu_t)dB_t + b(X_t, Y_t)dt; \quad X_0 = 1 \tag{3}$$

We present a method to detect when bubble pricing of a stock is occurring. It is far from perfect, and quite noisy, but with some tweaks it seems to work. Using TAQ data (Trade And Quote data) and a subsampling procedure from a large data set, we calculate bubble lifetimes on 3,500 stocks over a 13 year period, and are able to give the empirical distribution of the lifetimes of bubbles, which turns out to be a generalized gamma distribution, a distribution that arises naturally in survival analysis. We detected around 4 bubbles per stock, on average.

We give an argument that shows it can be considered natural that bubble lifetimes follow a generalized gamma distribution.

It would seem that this type of analysis could be useful for regulators estimating certain aspects of banking risk; for example if significant parts of a bank's portfolio are undergoing bubble pricing, then the risk is higher than if they are not. Moreover by knowing the bubble lifetime distribution, one can estimate the speed with which positions would need to be unwound.

References

[1] F. Biagini, H. Föllmer and S. Nedelcu (2014). *Shifting martingale measures and the slow birth of a bubble.* Finance and Stochastics **18**(2), 297–326.

[2] P.L. Lions and M. Musiela (2007). *Correlations and bounds for stochastic volatility models.* Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire **24**(1), 1–16.

[3] P. Protter (2013). *A Mathematical Theory of Bubbles.* F.E. Benth et al., Paris-Princeton Lectures on Mathematical Finance, Lecture Notes in Mathematics 2081, DOI 10.1007/978-3-319-00413-6 1, Springer, 1–108.

Bermudan options by simulation

Chris Rogers

The aim of this study is to devise numerical methods for dealing with very high-dimensional Bermudan-style derivatives. For such problems, we quickly see that we can at best hope for price bounds, and we can only use a simulation approach. We use the approach of Barraquand & Martineau [1] which proposes that the reward process should be treated as if it were Markovian, and then uses this to generate a stopping rule and hence a lower bound on the price. Using the dual approach introduced by Rogers [3] and Haugh & Kogan [2], this approximate Markov process leads us to hedging strategies, and upper bounds on the price. The methodology is generic, and is illustrated on eight examples of varying levels of difficulty. Run times are largely insensitive to dimension.

References

[1] J. Barraquand and D. Martineau (1995). *Numerical valuation of high dimensional multivariate American securities.* Journal of Financial and Quantitative Analysis **32**, 383–405.

[2] M. Haugh and L. Kogan (2004). *Pricing American options: a duality approach.* Operations Research **52**, 258–270.

[3] L.C.G. Rogers (2002). *Monte Carlo valuation of American options.* Mathematical Finance **12**, 271–286.

Risk bounds and partial dependence information

Ludger Rüschendorf

The subject of this talk is to describe some recent developments which aim to reduce value at risk (VaR)—bounds or TVaR—bounds for joint portfolios based on marginal information by including partial dependence information. For the case of marginal information several tools have been developed like dual bounds, mixing, rearrangement algorithm and convex ordering bounds to bound VaR in a sharp way. The available results in this area are described in Embrechts, Puccetti, Rüschendorf (2013) [4]. But it has turned out that the resulting bounds are typically to wide too be acceptable in practice.

A natural idea is to include higher order marginal information. In papers of Embrechts and Puccetti (2010) [3] and Puccetti and Rüschendorf (2012) [6] a system of improved bounds called reduced bounds was introduced which reduces the problem to a related problem with only marginal information. This reduced problem can therefore be solved by known techniques. The amount of improvement depends on the kind of the higher order marginals.

In Bernard, Rüschendorf and Vanduffel (2014) [1] an additional variance constraint of the form $\mathrm{Var}(S_n) \leq s^2$ is introduced. These lead to improved bounds which are simple to calculate. An extension of the RA-algorithm called ERA is given in the paper. As result one finds that for small enough variance bounds one gets a considerable reduction of the upper risk bounds. The reason for the reduction of the upper risk bounds is, that the variance restriction implies some global negative dependence constraint.

In the paper of Bignozzi, Puccetti and Rüschendorf (2015) [2] a series of positive (and negative) dependence constraints is introduced. Based on a classical result on improved Frechét bounds one gets in analytical form reduced VaR bounds. Examples show that **strong** forms of positive dependence restrictions yield considerable improvements of the lower risk while the weak positive orthant dependence (POD) assumption alone is not enough to improve VaR bounds. With the stronger PCD (positive cumulative dependence) or the WCS (weakly conditionally increasing in sequence) notions positive reductions of the DU-spread can be obtained.

A particular effective reduction results from the assumption that the risks are split into a number k of independent subgroups. This assumption is investigated in Puccetti, Rüschendorf, Small and Vanduffel (2015) [5]. In some real insurance portfolios our results are in concordance with practice but in contrast to the standard models in use in industry our bounds are based on reliable information.

A flexible reduction method uses as structural input a factor model of the form $X_i = f_i(Z_i, \varepsilon_i)$ with a systemic risk vector Z and without assuming the usual independence assumption on the individual risk ε_i. It is shown that a considerable reduction of the DU-spread is obtained over the whole range of positive and negative dependences generated by the systemic risk factor Z.

References

[1] C. Bernard, L. Rüschendorf and S. Vanduffel (2014). *Optimal claims with fixed payoff structure.* Journal of Applied Probability **51**A, 175–188.

[2] V. Bignozzi, G. Puccetti and L. Rüschendorf (2015). *Reducing model risk via positive and negative dependence assumptions.* Insurance Mathematics and Economics **61**(1), 17–26.

[3] P. Embrechts, G. Puccetti (2010). *Bounds for the sum of dependent risks having overlapping marginals.* Journal of Multivariate Analysis **101**(1), 177–190.

[4] P. Embrechts, G. Puccetti and L. Rüschendorf (2013). *Model uncertainty and VaR aggregation.* Journal of Banking & Finance **37**(8), 2750–2764.

[5] G. Puccetti, L. Rüschendorf, D. Small and S. Vanduffel (2015). *Reduction of Value-at-Risk bounds via independence and variance information.* doi: 10.2139/ssrn.2584860.

[6] G. Puccetti and L. Rüschendorf (2012). *Bounds for joint portfolios of dependent risks.* Statistics & Risk Modeling **29**(2), 107–132.

Semi-parametric models for multivariate extreme value distributions

Johan Segers

(joint work with Axel Bücher, Ivan Kojadinovic and Tom Rohmer)

1. Change-point detection

Given a sequence $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ of d-dimensional observations, change-point detection aims at testing

$$H_0 : \exists\, F \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have c.d.f. } F$$

against alternatives involving the nonconstancy of the c.d.f. Under H_0 and the assumption that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ have continuous marginal c.d.f.s $F_1, \ldots, F_d$, the common multivariate c.d.f. F can be written in a unique way as

$$F(\boldsymbol{x}) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

where the function $C : [0,1]^d \to [0,1]$ is a copula. It follows that H_0 can be rewritten as $H_{0,m} \cap H_{0,c}$, where

$$H_{0,m} : \exists\, F_1, \ldots, F_d \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have marginal c.d.f.s } F_1, \ldots, F_d,$$
$$H_{0,c} : \exists\, C \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have copula } C.$$

It is our aim to construct a new test for H_0 that is more powerful than its predecessors against alternatives that involve a change in the copula. The test is based on sequential empirical copula processes, but the crucial difference with respect to earlier proposals lies in the computation of the ranks. Whereas in [5] and subsequent papers, ranks are always computed with respect to the full sample, we propose to compute the ranks with respect to the relevant subsamples. In this way, the copulas of those subsamples are estimated more accurately, so that differences between copulas of disjoint subsamples are detected more quickly.

2. Test statistic

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be random vectors. For integers $1 \le k \le l \le n$, let $C_{k:l}$ be the empirical copula of the sample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$. Specifically,

$$C_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{k:l} \le \boldsymbol{u}),$$

for $\boldsymbol{u} \in [0,1]^d$, where

$$\hat{\boldsymbol{U}}_i^{k:l} = \frac{1}{l-k+1}(R_{i1}^{k:l}, \ldots, R_{id}^{k:l}), \qquad i \in \{k, \ldots, l\},$$

with $R_{ij}^{k:l} = \sum_{t=k}^{l} \mathbf{1}(X_{tj} \le X_{ij})$ the (maximal) rank of X_{ij} among $X_{kj}, \ldots, X_{lj}$. Note that the ranks are computed within the subsample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$ and not within the whole sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. By convention, $C_{k:l} = 0$ if $k > l$.

Write $\Delta = \{(s,t) \in [0,1]^2 : s \le t\}$. Let $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$ for $(s,t) \in \Delta$. Our test statistic is based on the difference process, $\mathbb{D}_n$, defined by

$$\mathbb{D}_n(s, \boldsymbol{u}) = \sqrt{n}\, \lambda_n(0,s)\, \lambda_n(s,1) \{C_{1:\lfloor ns \rfloor}(\boldsymbol{u}) - C_{\lfloor ns \rfloor + 1:n}(\boldsymbol{u})\}, \qquad (s, \boldsymbol{u}) \in [0,1]^{d+1}.$$

The test statistic for detecting changes in cross-sectional dependence is then

$$S_n = \max_{1 \le k \le n-1} \int_{[0,1]^d} \{\mathbb{D}_n(k/n, \boldsymbol{u})\}^2 \, \mathrm{d}C_{1:n}(\boldsymbol{u}).$$

The p-values are determined by the null distribution of S_n, whose large-sample limit is derived in [2]. To estimate the p-values from the data, a multiplier bootstrap method is proposed in that paper too. The limit distribution of the test statistic under the null hypothesis is unwieldy, but approximate p-values can still be computed via a multiplier resampling scheme. To deal with potential serial dependence, we make use of dependent multiplier sequences, an idea going back to [4] and revisited in [1].

A large-scale Monte Carlo simulation study confirms that the sensitivity of rank-based tests for the null hypothesis of a constant distribution against changes in cross-sectional dependence can be improved if ranks are computed with respect to relevant subsamples. In many cases, the test we propose achieves a higher power than the one proposed in [3].

References

[1] A. Bücher and I. Kojadinovic (2013). *A dependent multiplier bootstrap for the sequential empirical copula process under strong mixing.* Bernoulli, forthcoming.

[2] A. Bücher, I. Kojadinovic, T. Rohmer and J. Segers (2014). *Detecting changes in cross-sectional dependence in multivariate time series.* Journal of Multivariate Analysis**132**, 111–128.

[3] A. Bücher and M. Ruppert. *Consistent testing for a constant copula under strong mixing based on the tapered block multiplier technique.* J. Multivar. Anal. **116**, 208–229.

[4] P. Bühlmann (1993). *The blockwise bootstrap in time series and empirical processes.* PhD thesis, ETH Zürich. Diss. ETH No. 10354.

[5] B. Rémillard (2010). *Goodness-of-fit tests for copulas of multivariate time series.* Social Science Research Network, 1729982.

Fifty ways to hump consumption

Mogens Steffensen

In this talk, we discuss four different plausible explanations for hump-shaped demand for consumption that one can find in consumer data. The first explanation is a phenomenon that we here speak of as elasticity of biometric substitution. This aspect of preferences is formalized in Jensen and Steffensen (2015) who obtain a hump-shaped demand for consumption. The economic intuition is the following. A patient individual postpones consumption to later but at high ages an opposite effect kicks in. Then the high mortality forces him to consume while still being alive and this bends the consumption into a hump. The elasticity of biometric substitution is separated away from risk aversion and elasticity of intertemporal substitution and forms a demanding, but appealing, non-linear stochastic control problem.

The hump-shaped consumption is obtained by Jensen and Steffensen (2015) without over- or underpricing of insurance contracts. If we allow for mispricing of insurance contracts humps can be obtained under time- and state-additivity of preferences. In that case it just takes insurance to be overpriced in young years and underpriced in old years. At first glance, this sounds very unnatural but it may not be as stupid as it sounds. Since there is a demand for positive insurance benefits at young ages and a demand for negative insurance benefits at old ages (life annuities) this combination of under- and overpricing is exactly what one would expect. This mispricing of insurance contracts is our second explanation for humps.

The third explanation is the effect of investment in education studied in Munk et al. (2015a). They study a problem with Cobb-Douglas utility from consumption and leisure time net of working and education hours. Time spent on education pays off in terms of salary. For a patient individual consumption increases in younger years. In older years the education effect kicks in: education does no longer pay off close to retirement, the individual obtains utility from leisure and therefore lets the consumption decrease.

The fourth effect was suggested by Munk et al. (2015b). An impatient individual is considered. Without habit formation, this individual demands decreasing consumption. Adding habit formation to his preferences, however, scares him away from consuming too much at the beginning in order not to get trapped at a too high level of consumption. This effect vanishes with time and the combination of the two effects can produce a consumption hump.

Why are four explanations enough to explain the title of the talk? Well, Paul Simon pointed out, after giving four ways to leave your lover, that there must be 50. So, there must be ... 50 ways to hump consumption.

References

[1] N.R. Jensen and M. Steffensen (2015). *Personal finance and life insurance under separation of risk aversion and elasticity of substitution.* Insurance: Mathematics and Economics **62**, 28–41

[2] H. Kraft, C. Munk, F.T. Seifried and M. Steffensen (2015). *Consumption and wage humps in a life-cycle model with education.* Available at SSRN: http://ssrn.com/abstract=2448191 or http://dx.doi.org/10.2139/ssrn.2448191 (February 24, 2015),

[3] H. Kraft, C. Munk, F.T. Seifried and S. Wagner (2015b). *Consumption habits and humps.* SAFE Working Paper **15, available at SSRN:** http://ssrn.com/abstract=2259613 **or** http://dx.doi.org/10.2139/ssrn.2259613 **(July 10, 2015).**

Max-stable random sup-measures with comonotonic tail dependence

Kirstin Strokorb

(joint work with Ilya Molchanov)

In this project [7], we reveal some connections between extreme value theory (max-stable processes), stochastic geometry (random sets) and the theory of utility functions or risk measures (comonotonic additivity).

Max-stable random sup-measures can be seen as a generalization of max-stable processes or, if endowed with an appropriate topology, they are equivalent to max-stable processes with upper semi-continuous sample paths [8, 9, 15]. Random sup-measures with independent values on disjoint sets are called completely random or having independent peaks [8, 13]. They are well understood including the corresponding integration theory that relies on the concept of the extremal integral [13]. The distribution of a max-stable completely random sup-measure is fully characterized by its control measure, similarly to the situation with conventional α-stable completely random measures studied in details in [11].

In this work we go beyond the completely random case and study properties of general max-stable random sup-measures in relation to their tail dependence functionals using the extremal integral from [4]. It is well-kown that, under very mild conditions, max-stable processes allow for a Poisson process representation [1, 6, 16]. We show that the same holds true for the max-stable random sup-measures and, motivated by [14], we single out a sub-family (the sub-family of TM sup-measures) that arises from a very natural choice of the intensity of the underlying Poisson process. It turns out that this choice is equivalent to a comonotonicity property that is well-known in the theory of risk measures [2, 12].

Based on [3, 5], we obtain dual representations for the tail dependence functionals of the general stable sup-measures and TM sup-measures, which is useful in order to derive a stochastic dominance property of TM sup-measures. With each general stable sup-measure it is possible to associate several natural TM sup-measures. This also leads to a streamlined extension of some known properties of the argmax-set in continuous choice models [10].

References

[1] L. de Haan (1984). *A spectral representation for max-stable processes.* Annals of Probability. **12**(4), 1194–1204.

[2] H. Föllmer and A. Schied (2004). *Stochastic Finance. An Introduction in Discrete Time.* De Gruyter, Berlin, 2nd edition.

[3] B. Fuglede (1977). *Capacity as a sublinear functional generalizing an integral.* Mat.-Fys. Medd. Danske Vid. Selsk. **38**(7), 44.
[4] B. Gerritse (1996). *Varadhan's theorem for capacities.* Commentationes Mathematicae Universitatis Carolinae **37**, 667–690.
[5] S. Graf (1980) *A Radon-Nikodym theorem for capacities.* Journal für die reine und angewandte Mathematik **320**, 192–214.
[6] Z. Kabluchko (2009). *Spectral representations of sum- and max-stable processes.* Extremes **12**(4), 401–424.
[7] I. Molchanov and K. Strokorb. *Max-stable random sup-measures with comonotonic tail dependence.* Technical report, `http://arXiv.org/abs/1507.03476`.
[8] T. Norberg (1986). *Random capacities and their distributions.* Probability Theory and Related Fields **73**(2), 281–297.
[9] G.L. O'Brien, P. J. J.F. Torfs and W. Vervaat (1990). *Stationary self-similar extremal processes.* Probability Theory and Related Fields **87**(1), 97–119.
[10] S. I. Resnick and R. Roy (1991). *Random USC functions, max-stable processes and continuous choice.* Annals of Applied Probability **1**(2), 267–292.
[11] G. Samorodnitsky and M. S. Taqqu (1994). *Stable non-Gaussian Random Processes.* Chapman & Hall, New York.
[12] D. Schmeidler (1986). *Integral representation without additivity.* Proceedings of the American Mathematical Society **97**, 255–261.
[13] S. Stoev and M.S. Taqqu (2005). *Extremal stochastic integrals: a parallel between max-stable processes and α-stable processes.* Extremes **8**, 237–266.
[14] K. Strokorb and M. Schlather (2015). *An exceptional max-stable process fully parametrized by its extremal coefficients.* Bernoulli **21**, 276–302.
[15] W. Vervaat (1997). *Random upper semicontinuous functions and extremal processes.* In *Probability and Lattices*, edited by W. Vervaat and H. Holwerda, editors, volume **110** of *CWI Tracts*, 1–56. CWI, Amsterdam.
[16] Y. Wang and S. A. Stoev (2010). *On the structure and representations of max-stable processes.* Advances in Applied Probability, **42**(3), 855–877.

Limit results for bivariate distributions using polar representations: a review of recent developments

Miriam Isabel Seifert

An effective approach for the analysis of multivariate extreme values is to investigate the limit behavior of random vectors given that one of the vector components becomes large. For this purpose, the *conditional extreme value (CEV) models* have been proposed by Heffernan and Tawn (2004) and further developed by Heffernan and Resnick (2007), Das and Resnick (2011), Resnick and Zeber (2014). In the CEV context, bivariate distributions of (X, Y) are studied such that

$$\lim_{x\to\infty} P(X \le \alpha(x) + \beta(x)\xi,\ Y \le \gamma(x) + \delta(x)\zeta \mid X > x) \;=\; G(\xi,\zeta) \tag{1}$$

holds with suitable normalizing functions α, β, γ, δ and a non-degenerate distribution function G.

Following this CEV approach we discuss limit results for important families of bivariate distributions. We start with the popular elliptical random vectors which can be represented as

$$(X, Y) \;=\; R\cdot(\cos T, \rho\cos T + \sqrt{1-\rho^2}\sin T) \tag{2}$$

with stochastically independent polar components $R \geq 0$ and $T \in (-\pi, \pi]$, where T is uniformly distributed and $\rho \in (-1, 1)$.

Assume R being in the Gumbel max-domain of attraction with infinite right endpoint which is equivalent to R being of type $\Gamma(\psi)$ with some auxiliary function ψ, i.e. for all $z \in \mathbb{R}$ holds

$$\lim_{x\to\infty} \frac{P\{R > x + z\psi(x)\}}{P\{R > x\}} \;=\; \mathrm{e}^{-z}\,. \tag{3}$$

For this case, Berman (1983) obtained the following conditional limit result for $\xi > 0,\ \zeta \in \mathbb{R}$:

$$\lim_{x\to\infty} P(X \leq x\ + \psi(x)\xi,\ \ Y \leq \rho x\ + \delta(x)\zeta \mid X > x) \;=\; \left(1 - \mathrm{e}^{-\xi}\right) \cdot K\left(\zeta\right) \tag{4}$$

$$\text{with} \quad \delta(x) \;=\; x \cdot \left(\left(1 - \rho^2\right) \cdot \frac{\psi(x)}{x} \right)^{1/2}, \quad K\left(\zeta\right) \;=\; \Phi\left(\zeta\right).$$

The class $\Gamma(\psi)$ covers light-tailed distributions like the normal or exponential ones as well as mildly heavy-tailed distributions like the log-normal.

Abdous et al. (2005) complemented this limit result of Berman by investigating the conditional limit behavior of elliptical random vectors with heavy-tailed R in the Fréchet max-domain of attraction, where the limit distribution $G(\xi, \zeta)$ has no longer a product form $G_1(\xi) \times G_2(\zeta)$ of its marginals.

Next we consider different directions to extend the class of elliptical distributions in the context of the CEV modeling. One approach is to weaken the assumptions on the level lines of the joint density of (X, Y) from elliptical to more general ones. For this purpose Balkema and Embrechts (2007) introduced the class of rotund-exponential distributions characterized by homothetic densities with level lines possessing a positive curvature at every point and deduced conditional limit theorems.

In order to generalize the limit result of Berman we follow another approach which is based on the idea to extend the elliptical representation in (2) for random vectors (X, Y) to even more broad *polar representations*

$$(X, Y) \;=\; R \cdot (u(T), v(T)) \tag{5}$$

with stochastically independent *polar components* $R \in [0, \infty)$ of type $\Gamma(\psi)$ and T with a positive, continuous density on some closed interval in $\mathbb{R}$. The level curves are characterized by quite arbitrary coordinate functions u and v, where u takes its unique global maximum $u_{\max}$ at $t = t_0$. We describe the geometry of the curve $(u(t), v(t))$ in some neighborhood of $t = t_0$ by function $l = u_{\max} - u$ displaying the horizontal distance to the vertical ray $\{x = u_{\max}\}$ and assume that l is regularly varying at $t = t_0$ with some variation index $\kappa > 0$. Hence, κ is the local curve order of the level curves near the ray $y = \rho x$ with $\rho := v(t_0)$.

Under polar representation in (5) the result of Berman for the elliptical case $\kappa = 2$ has been generalized by Fougères and Soulier (2010), who proved that for curve orders $\kappa > 1$ the statement in (4) holds with

$$\delta(x) \ = \ x \cdot C \cdot \left(\frac{\psi(x)}{x}\right)^{1/\kappa}, \quad K(\zeta) \ = \ c \cdot \int_{-\infty}^{\zeta} \exp\big(-|s|^{\kappa}/\kappa\big)\mathrm{d}s \ .$$

The question arises whether conditional limit statements could be obtained for polar random vectors $(X, Y) = R \cdot (u(T), v(T))$ for arbitrary values $\kappa > 0$.
We investigate (Seifert 2014) how the restriction $\kappa > 1$ can be dropped so that the level curves do not have to be convex any longer. For $\kappa = 1$ they form "edges", for $0 < \kappa < 1$ they form "cusps" located on the ray $y = \rho x$. It turns out that there is a fundamental difference between the cases $\kappa < 1$ and $\kappa > 1$; we explain the consequences of these distinctions and analyze them geometrically.

We obtain the following important results (Seifert 2014). In the generic case $\rho \neq 0$, the conventional CEV method as in (1) leads to a degenerate limit G for $\kappa < 1$. However, using *random norming*, where functions bounding Y from above are not evaluated at the threshold value x but at the actual value of X, we manage to obtain a non-degenerate limit theorem for polar random vectors with *arbitrary* curve order $\kappa > 0$ and $\rho \in \mathbb{R}$:

$$(6) \qquad \lim_{x\to\infty} P\big(X \le x + \psi(x)\xi, \ Y \le \rho X + X \cdot C \cdot (\psi(X)/X)^{\,1/\kappa}\,\zeta \ \big| \ X > x\big)$$
$$= \ c \cdot \big(1 - \mathrm{e}^{-\xi}\big) \cdot \int_{-\infty}^{\zeta} \exp\left(-|s|^{\kappa}/\kappa\right)\mathrm{d}s$$

for $\xi > 0, \ \zeta \in \mathbb{R}$. The method of random norming was implicitly introduced by Heffernan and Tawn (2004) in their extrapolation algorithm for extreme values and further investigated by Heffernan and Resnick (2007).
Thus, applying random norming is the way to obtain non-degenerate conditional limit statements for the whole family of distributions with polar representations, allowing quite different forms of the level curves and, hence, permitting a lot of freedom for describing the asymptotic behavior of random vectors (X, Y).

Further generalizations to non-homothetic densities are deduced by Balkema and Embrechts (2007): they introduced a class $\mathcal{L}$ of functions such that the rotund-exponential density f_0 of (X, Y) can be multiplied by them without changing the asymptotics.
Alternatively, we investigate (Seifert 2015) how the independence assumption of radial component R and angular component T for $(X, Y) = R \cdot (u(T), v(T))$ can be weakened. We propose a novel dependence measure and present convenient criteria for validity of limit theorems which possess geometrical meaning. Such results also verify stability of the previously discussed limit results for a certain degree of dependence.

References

[1] B. Abdous, A.-L. Fougères and K. Ghoudi (2005). *Extreme behavior for bivariate elliptical distributions.* Canadian Journal of Statistics **33**(3), 317–334.

[2] G. Balkema and P. Embrechts (2007). *High risk scenarios and extremes. A geometric approach.* European Mathematical Society, Zurich Lectures in Advanced Mathematics.
[3] S.M. Berman (1983). *Sojourns and extremes of Fourier sums and series with random coefficients.* Stochastic Processes and their Applications 15(3), 213–238.
[4] B. Das and S.I. Resnick (2011). *Conditioning on an extreme component: model consistency and regular variation on cones.* Bernoulli **17**(1), 226–252.
[5] A.-L. Fougères and P. Soulier (2010). *Limit conditional distributions for bivariate vectors with polar representation. Stochastic Models* **26**(1), 54–77.
[6] J.E. Heffernan and S.I. Resnick (2007). *Limit laws for random vectors with an extreme component.* Annals of Applied Probability **17**(2), 537–571.
[7] J.E. Heffernan and J.A. Tawn (2004). *A conditional approach for multivariate extreme values.* Journal of the Royal Statistical Society (Series B) 66(3), 497–546.
[8] S.I. Resnick and D. Zeber (2014). *Transition kernels and the conditional extreme value model.* Extremes **17**(2), 263–287.
[9] M.I. Seifert (2014). *On conditional extreme values of random vectors with polar representation.* Extremes **17**(2), 193–219.
[10] M.I. Seifert (2015). *Weakening the independence assumption on polar components: limit theorems for generalized elliptical distributions.* To appear in *Journal of Applied Probability.*

Jump regressions

George Tauchen
(joint work with Jia Li and Viktor Todorov)

We develop in [1] statistical tools for studying jump dependence of two processes from high-frequency observations on a fixed time interval. In this context, only segments of data around a few outlying observations are informative for the inference. The underlying theory draws on [2] who characterize the behavior of the asset returns across a discrete time interval containing a jump move in the process. We derive an asymptotically valid test for stability of a linear jump relation over regions of the jump size domain. The test has power against general forms of nonlinearity in the jump dependence as well as temporal instabilities. We further propose an optimal estimator for the linear jump regression model that is formed by optimally weighting the detected jumps with weights based on the diffusive volatility around the jump times. We derive the asymptotic limit of the estimator, a semiparametric lower efficiency bound for the linear jump regression, and show that our estimator attains the latter. A higher-order asymptotic expansion for the optimal estimator further allows for finite-sample refinements. In an empirical application, we use the developed inference techniques to test the stability (in time and jump size) of market jump betas.

References

[1] J. Li, G. Tauchen and V. Todorov (2015). *Jump regressions.* Duke University and Northwestern University, technical report.
[2] J. Jacod and P. Protter (2012). *Discretization of Processes.* Springer.

Testing and modelling the unconditional variance component in multiplicative time-varying GARCH models

Timo Teräsvirta

This presentation is about extending the standard GARCH model to the case where the underlying volatility process is no longer stationary. This extension consists of completing the standard model by a multiplicative nonstationary component. The nonstationary component is a positive-valued function of time (or rescaled time). The parameterisation makes it possible to test constancy of the unconditional variance (stationarity of the GARCH model) against the nonstationary GARCH. It also gives rise to specification issues that I have discussed and applied in my work with Cristina Amado.

One of the specification issues is finding the 'right' parametric form for the nonstationary component. In Amado and Teräsvirta (in press) the test of constancy of the unconditional variance is a misspecification test, that is, the test is carried out *after* the GARCH component has been specified and estimated. The resulting test is size-distorted even in rather large samples. In this presentation I turn this around and test the null hypothesis *before* estimating the GARCH component. The problem of neglected conditional heteroskedasticity appears, but the size distortion turns out to be rather small.

This can be taken a step further. The unconditional variance component is specified using sequential testing, see Luukkonen et al. (1988) for the idea for the test. The whole specification process, that is, the sequence of tests, can now be carried out without assuming any conditional heteroskedasticity. This provides initial estimates for final estimation in which the GARCH component is estimated by maximum likelihood jointly with the unconditional variance component. For asymptotic properties of maximum likelihood estimators, see Amado and Teräsvirta (2013). In the presentation I give an example of how this works. The results are encouraging.

One can replace time by an exogenous random variable and use the same specification strategy. Another idea that seems worth considering is that there may be more than one multiplicative component in the model. For example, one may be a function of time whereas the other one is a positive-valued function of an exogenous random variable. How to test for the second multiplicative component and how the specification strategy would work in this situation will be investigated in the future.

References

[1] C. Amado and T. Teräsvirta (2013). *Modelling volatility by variance decomposition.* Journal of Econometrics **175**, 142–153.

[2] C. Amado and T. Teräsvirta (in press). *Specification and testing of multiplicative time-varying GARCH models with applications.* Econometric Reviews.

[3] R. Luukkonen, P. Saikkonen and T. Teräsvirta (1988). *Testing linearity against smooth transition autoregressive models.* Biometrika **75**, 491–499.

[4] A. Silvennoinen and T. Teräsvirta (2015). *Testing constancy of unconditional variance in volatility models by misspecification and specification tests.* CREATES Research Paper, Aarhus University.

Testing for time-varying jump activity for pure jump semimartingales

Victor Todorov

In this paper we propose a test for deciding whether the jump activity index of a discretely-observed Ito semimartingale of pure-jump type (i.e., one without a diffusion) varies over a fixed interval of time. The asymptotic setting is based on observations within a fixed time interval with mesh of the observation grid shrinking to zero. The test is derived for semimartingales whose "spot" jump compensator around zero is like that of a stable process, but importantly the stability index can vary over the time interval. The test is based on forming a sequence of local estimators of the jump activity over blocks of shrinking time span and contrasting their variability around a global activity estimator based on the whole data set. The local and global jump activity estimates are constructed from the real part of the empirical characteristic function of the increments of the process scaled by local power variations. We derive the asymptotic distribution of the test statistic under the null hypothesis of constant jump activity and show that the test has asymptotic power of one against fixed alternatives of processes with time-varying jump activity.

References

[1] V. Todorov (2015). *Jump activity estimation for pure-jump semimartingales via self-normalized statistics.* Annals of Statistics **43**, 1831–1864.

Parsimony inducing priors for large scale state-space models with applications in modeling high-dimensional volatilities

Ruey Tsay
(joint work with Hedibert Lopes and Robert McCulloch)

State-space models are commonly used in the engineering, economic, and statistical literatures. They are flexible and encompass many well-known statistical models, including random coefficient autoregressive models and dynamic factor models. Bayesian analysis of state-space models has attracted much interest in recent years. However, for large scale models, prior specification becomes a challenging issue in Bayesian inference. In this paper, we propose a flexible prior for state-space models. The proposed prior is a mixture of four commonly entertained models, yet achieves parsimony in high-dimensional systems. Here "parsimony" is represented by the idea that in a large system, some states may not be time-varying. Simulation and simple examples are used throughout to demonstrate the performance of the proposed prior. As an application, we consider the time-varying conditional covariance matrices of daily log returns of 94 components of the S&P

100 index, leading to a state-space model with 94×95/2=4,465 time-varying states. Our model for this large system enables us to use parallel computing.

References

[1] H.F. Lopes, R.E. McCulloch and R.S. Tsay (2014). *Parsimony inducing priors for large-scale state-space models.* University of Chicago, Booth School of Business, working paper.

Aggregation of risk measures

Ruodu Wang

Modeling inter-dependence among multiple risks often faces statistical as well as modeling challenges, with considerable uncertainty arising naturally. To deal with the uncertainty in dependence of multivariate models, the field of risk aggregation with dependence uncertainty has been greatly developed in the past few years. The main object of interest is the set $\mathcal{D}_n$ of possible distributions of risk aggregation with given marginal information and arbitrary dependence structure. More precisely, for given univariate distributions $F_1, \dots, F_n$, define

$$\mathcal{D}_n(F_1, \dots, F_n) = \{\text{cdf of } X_1 + \cdots + X_n : X_i \in L^0,\ X_i \sim F_i,\ i = 1, \dots, n\}.$$

A direct characterization of $\mathcal{D}_n$ for general marginal distributions $F_1, \dots, F_n$ is unavailable at the moment, and many open questions are found around it. We discuss two representative concrete mathematical questions within this framework: joint mixability and extreme values of risk measures.

A tuple of distribution functions $(F_1, \dots, F_n)$ is *jointly mixable* if $\mathcal{D}_n$ contains a point mass. An analytical method to verify this property is in general an open question. For some particular classes of distributions, joint mixability can be analytically characterized; see [3] for details. Joint mixability is naturally linked to many optimization problems in finance and operations research.

Extreme values of risk measures under uncertainty are of particular interest in risk management practice, and there are a lot of research in this direction; see [1] for the case of VaR. It is not clear how to calculate numerically or analytically the extreme values of a risk measure all possible risks with distributions in $\mathcal{D}_n$. However, as $n \to \infty$, it is shown that for a distortion or convex risk measure ρ, its worst-case value over is asymptotically equivalent to that of another risk measure ρ^*, which is coherent and hence easy to calculate; see [4] for details. It is further obtained in [2] that the dependence-uncertainty spread (the worst-case value minus the best-case value over all possible models) of a Value-at-Risk is generally larger than that of a corresponding Expected Shortfall, and this supports the use of an Expected Shortfall in risk aggregation.

References

[1] P. Embrechts, G. Puccetti, L. Rüschendorf, R. Wang and A. Beleraj (2014). *An academic response to Basel 3.5.* Risks **2**(1), 25–48.
[2] P. Embrechts, B. Wang and R. Wang (2015). Aggregation-robustness and model uncertainty of regulatory risk measures. *Finance and Stochastics*, forthcoming. DOI: 10.1007/s00780-015-0273-z
[3] B. Wang and R. Wang (2015). *Joint mixability. Mathematics of Operations Research*, forthcoming.
[4] R. Wang, V. Bignozzi and A. Tsakanas (2015). *How superadditive can a risk measure be?* SIAM Journal on Financial Mathematics **6**, 776–803.

Bounds on asymptotic Value-at-Risk for multivariate regularly varying random vectors

Robert Alohimakalani Yuen
(joint work with Stilian Stoev and Dan Cooley)

The Value-at-Risk of a random variable $X \in \mathbb{R}$ at the level $\alpha \in (0,1)$, denoted $\mathrm{VaR}_\alpha(X)$ is defined as

$$\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) \geq \alpha\}.$$

Now consider a portfolio $\{1, \dots, d\}$ of dependent losses $\boldsymbol{X} = (X_1, X_2, \dots, X_d)^\top \in \mathbb{R}^d$, and the Value-at-Risk for the sum of losses $\mathrm{VaR}_\alpha(S)$, where $S := X_1 + X_2 + \cdots + X_d$. Here it is essential to account for tail dependence in the components of $\boldsymbol{X}$ because regulatory guidelines typically prescribe $\alpha \geq .99$. Hence, α close to the value 1 is of primary interest. Multivariate regular variation is a natural framework for characterizing $\mathrm{VaR}_\alpha(S)$ when $\alpha \approx 1$. A vector $\boldsymbol{X} \in \mathbb{R}^d_+$ is multivariate regularly varying with index $-1/\xi$ if there exists a function $h(t)$ with $\lim_{t\to\infty} h(t) = \infty$ such that $\lim_{t\to\infty} t\mathbb{P}(X_1 > sh(t)) = s^{-1/\xi}$ for all $s > 0$, a constant $\rho > 0$, and a measure H on $\mathbb{S}^{d-1}_+ = \{\boldsymbol{u} \in \mathbb{R}^d_+ : u_1 + u_2 + \cdots + u_d = 1\}$ with finite mass $H(\mathbb{S}^{d-1}_+) = d$, called the *spectral measure,* such that for all $s > 0$

$$\lim_{t\to\infty} t\mathbb{P}\left(S > sh(t), \frac{\boldsymbol{X}}{S} \in A\right) = \rho s^{-1/\xi} H(A)/d, \tag{1}$$

for $A \subset \mathbb{S}^{d-1}_+$, a continuity set of H. If the above holds then we say $\boldsymbol{X} \in \mathrm{MRV}^d_+(-1/\xi)$ and it readily follows that

$$\lim_{s\to\infty} \frac{\mathbb{P}(S > s)}{\mathbb{P}(X_1 > s)} = \rho.$$

Furthermore, if $\boldsymbol{X} \in \mathrm{MRV}^d_+(-1/\xi)$, then

$$\lim_{\alpha \nearrow 1} \frac{\mathrm{VaR}_\alpha(S)}{\mathrm{VaR}_\alpha(X_1)} = \rho^\xi.$$

The value ρ determines the extreme Value-at-Risk (expressed as a limit) for the sum of losses $S = X_1 + X_2 + \cdots + X_d$, normalized by the common marginal. It was shown in [1] that $\boldsymbol{X} \in \mathrm{MRV}_+^d(-1/\xi)$ implies

$$\rho \equiv \rho(H,\xi) := \int_{\mathbb{S}_+^{d-1}} (u_1^\xi + u_2^\xi + \cdots + u_d^\xi)^{1/\xi} H(d\boldsymbol{u}) \tag{2}$$

where H is the spectral measure in (1) which, theoretically, could be any finite measure on $\mathbb{S}_+^{d-1}$ satisfying marginal equality constraints

$$1 = \int_{\mathbb{S}_+^{d-1}} u_j H(d\boldsymbol{u}),\ j = 1, \dots, d. \tag{3}$$

Well known universal bounds on the value of ρ are given by

$$d \leq \rho(H,\xi) \leq d^{1/\xi} \quad \xi \leq 1 \tag{4}$$

$$d^{1/\xi} \leq \rho(H,\xi) \leq d \quad \xi \geq 1, \tag{5}$$

(see e.g. Corollary 4.2 of [3]). Observe that for $\xi = 1$ we have $\rho = d$, regardless of the form of H. Otherwise, $\rho = d$ corresponds to mutual independence and $\rho = d^{1/\xi}$ corresponds to complete tail dependence of components of the vector $\boldsymbol{X}$. The fact that H is itself an infinite dimensional parameter makes fully characterizing tail dependence a difficult problem [2]. In contrast, one can estimate various finite dimensional functionals which summarize the dependence of $\boldsymbol{X}$. One set of functionals is the *extremal coefficients* (see e.g [4]). Let H be the spectral measure of $\boldsymbol{X} \in \mathrm{MRV}_+^d(-1/\xi)$ and J a non-empty subset of $\{1, \dots, d\}$. The J^{th} extremal coefficient of with respect to H is

$$\vartheta_H(J) := \int_{\mathbb{S}_+^{d-1}} \max_{j\in J}\{u_j\} H(d\boldsymbol{u}).$$

Extremal coefficients alone do not fully characterize the spectral measure and the extent to which additional information given by extremal coefficients constrain the range of possible ρ was not previously known. Our objective is to determine sharp bounds on the value of ρ when obtaining full or partial knowledge of the extremal coefficients. That is, we want to determine exactly the interval

$$(P^*) \qquad \left(\inf_H \rho(H,\xi), \sup_H \rho(H,\xi)\right) \tag{6}$$

$$\text{subject to:} \qquad \int_{\mathbb{S}_+^{d-1}} \max_{j\in J}\{u_j\} H(d\boldsymbol{u}) = c_J,\ \text{for all } J \in \mathcal{J}, \tag{7}$$

where the supremum and infimum are taken over all finite measures on $\mathbb{S}_+^{d-1}$, and $\mathcal{J}$ is a given collection of subsets of $\{1, 2, \dots, d\}$.

In this talk, we characterize the solution to the problem (P^*). We show that the inf and sup in (6) are in fact attained by discrete measures that are supported on a finite set of atoms. In each case, the number of atoms is not more than the number of constraints in (7).

References

[1] P. Barbe, A. Fougeres and C. Genest (2006). *On the tail behavior of sums of dependent risks.* Astin Bull **36**, 361–373.
[2] J.H. Einmahl, A. Krajina and J. Segers (2012). *An M-estimator for tail dependence in arbitrary dimension.* Annals of Statistics **40**(3), 1764–1793. .
[3] P. Embrechts, D.D. Lambrigger and M.V. Wüthrich (2009). *Multivariate extremes and the aggregation of dependent risks: examples and counter-examples.* Extremes **12**, 107–127.
[4] R. Smith (1990). *Max-stable processes and spatial extremes.* Unpublished Manuscript.

The best time to leave a casino

Xun Yu Zhou
(joint work with Xue Dong He, Sang Hu, Jan Obłój)

We consider the dynamic casino gambling model initially proposed by [1] and study the optimal stopping strategy of a pre-committing gambler with cumulative prospect theory (CPT) preferences. We develop a systematic and analytical approach to finding the gambler's optimal strategy. We illustrate how the strategies computed in [1] can be strictly improved by reviewing the betting history or by tossing an independent coin, and we explain that the improvement generated by using randomized strategies results from the lack of quasi-convexity of CPT preferences. Finally, we show that any path-dependent strategy is equivalent to a randomization of path-independent strategies.

References

[1] N. Barberis (2012). *A model of casino gambling.* Management Science **58**, 35–51.

How is elicitability relevant for backtesting?

Johanna F. Ziegel
(joint work with Tobias Fissler and Tilmann Gneiting)

Independently, Weber [7] and Gneiting [6] have shown that Expected Shortfall (ES) is not elicitable in contrast to Value at Risk (VaR). Roughly, elicitability of a risk measure means that it can be obtained as the minimizer of an expected loss function. This negative result continues to hold for all spectral risk measures (except for the mean) and the only coherent risk measures that are elicitable are certain expectiles. However, we were able to show recently that ES is jointly elicitable with VaR, and, more generally, a large class of spectral risk measures is elicitable of higher order [4].

There is little debate that elicitability is a useful property for model selection, estimation, generalized regression, forecast comparison, and forecast ranking. But the non-elicitability of ES has lead to a lively debate about the relevance of elicitability for backtesting [1, 2, 3]. Contributing to this discussion, we would like to clarify that elicitability is not important for the traditional approach to backtesting. However, we argue that elicitability is crucial to achieve the objectives of backtesting [5]. We illustrate the proposed approach for VaR and ES jointly and for VaR alone.

References

[1] C. Acerbi and B. Székely (2014), *Back-testing expected shortfall.* Risk, December.

[2] M. Davis (2014). *Consistency of internal risk measure estimates.* Preprint, arXiv:1410.4382.

[3] S. Emmer, M. Kratz and D. Tasche (2015). *What is the best risk measure in practice? A comparison of standard measures.* Preprint , arXiv:1312.1645v4.

[4] T. Fissler and J. F. Ziegel (2015). *Higher order elicitability and Osband's principle.* Preprint (2015), arXiv:1503.08123.

[5] T. Fissler, J.F. Ziegel and T. Gneiting (2015). *Expected shortfall is jointly elicitable with value at risk – implications for backtesting.* Preprint, arXiv:1507.00244.

[6] T. Gneiting (2011). *Making and evaluating point forecasts.* Journal of the American Statistical Association **106**, 746–762.

[7] S. Weber (2006). *Distribution-invariant risk measures, information, and dynamic consistency.* Mathematical Finance **16**, 419–441.

Reporter: Thomas Mikosch